Directional statistics for WIMP direct detection II: 2-d read-out

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The direction dependence of the WIMP direct detection rate provides a powerful tool for distinguishing a WIMP signal from possible backgrounds. We study the the number of events required to discriminate a WIMP signal from an isotropic background for a detector with 2-d read-out using non-parametric circular statistics. We also examine the number of events needed to i) detect a deviation from rotational symmetry, due to flattening of the Milky Way halo and ii) detect a deviation in the mean direction due to a tidal stream. If the senses of the recoils are measured then of order 20-70 events (depending on the plane of the 2-d read out and the detector location) will be sufficient to reject isotropy of the raw recoil angles at 90% confidence. If the senses can not be measured these number increase by roughly two orders of magnitude (compared with an increase of one order of magnitude for the case of full 3-d read-out). The distributions of the reduced angles, with the (time dependent) direction of solar motion subtracted, are far more anisotropic, however, and if the isotropy tests are applied to these angles then the numbers of events required are similar to the case of 3-d read-out. A deviation from rotational symmetry will only be detectable if the Milky Way halo is significantly flattened. The deviation in the mean direction due to a tidal stream is potentially detectable, however, depending on the density and direction of the stream. The meridian plane (which contains the Earth's spin axis) is, for all detector locations, the optimum read-out plane for rejecting isotropy. However read-out in this plane can not be used for detecting flattening of the Milky Way halo or a stream with direction perpendicular to the galactic plane. In these cases the optimum read-out plane depends on the detector location.

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I. INTRODUCTION

Weakly Interacting Massive Particle (WIMP) direct detection experiments aim to directly detect non-baryonic dark matter via the elastic scattering of WIMPs on detector nuclei, and are presently reaching the sensitivity required to detect neutralinos (the lightest supersymmetric particle and an excellent WIMP candidate). The direction dependence of the event rate due to the Earth's motion [1] provides a powerful WIMP 'smoking gun'; a directional detector needs only of order ten events to differentiate a WIMP signal from isotropic backgrounds [2, 3, 4]. Designing a detector capable of measuring the directions of sub-100 keV nuclear recoils is a considerable challenge, however. Low pressure gas time projection chambers (TPCs), such as DRIFT (**D**irectional **R**ecoil **I**dentification **F**rom **T**racks) [5, 6] and NEWAGE [7], seem to offer the best prospects for a workable detector.

In paper I [4] we studied the number of events required to reject isotropy (and hence detect a WIMP signal) and reject rotational symmetry (and detect flattening of the Milky Way halo) for a range of observationally motivated halo models, taking into account the detector response. We also calculated the number of events required to detect a deviation in the mean direction from the direction of solar motion due to a tidal stream. We found that if the senses (i.e the signs) of the recoil vectors are known then of order ten events will be sufficient to distinguish a WIMP signal from an isotropic background for all of the halo models considered, with the uncertainties in reconstructing the recoil direction only mildly increasing the required number of events. If the senses of the recoils are not known then the number of events required is an order of magnitude larger, with a large variation between halo models, and the recoil resolution is now an important factor. The rotational symmetry test required of order a thousand events to distinguish between spherical and significantly triaxial halos, however a deviation of the peak recoil direction from the direction of the solar motion due to a tidal stream could be detected with of order a hundred events, regardless of whether the sense of the recoils is known. While technologies for 3-d TPC readouts with sufficient resolution to reconstruct sub-100 keV recoils in 3-d exist [7, 8], the cost and technological challenge of scaling these up to large, low background WIMP detectors is considerable. Therefore in this paper we repeat our analysis for a detector with less complex 2-d read-out to assess the effects this would have on the detection potential. Our goals are to assess the capabilities of the next generation of detectors and present analysis techniques which can be applied to real data (taking into account experimental practicalities/limitations and the uncertainty in the underlying WIMP distribution).

In Sec. II we briefly review our calculation of the nuclear recoil spectrum, including the modelling of the Milky Way halo. In Sec. III we then apply an array of statistical tests aimed at probing the isotropy (Sec. III A), rotational symmetry (Sec. III B) and mean direction (Sec. III C) of a putative WIMP directional sig-

nal, before concluding with discussion of our results in Sec. IV. In Appendix A we outline the circular statistics used.

II. CALCULATING THE NUCLEAR RECOIL SPECTRUM

The nuclear recoil spectrum depends sensitively on the (unknown) local WIMP velocity distribution. Observations and simulations of halos indicate that it is likely that the Milky Way (MW) halo is triaxial, anisotropic and contains substructure (see Paper I [4] for discussion and references) and these properties can lead to interesting features in the recoil distribution spectrum [2, 4, 9]. A generic feature of triaxial halo models is a flattening of the recoil distribution towards the Galactic plane, so that the recoil distribution is not symmetric about the direction of motion of the Sun, $(l_{\odot}, b_{\odot})$. WIMPs from a tidal stream, with velocity dispersion small compared with its bulk velocity, produce a recoil distribution peaked in the hemisphere whose pole points in the direction of the stream velocity. The net (stream plus smooth background WIMP distribution) peak direction depends on the direction of the stream and the fraction of the local density that it contributes.

We consider three fiducial halo models (selected from the twelve considered in paper I) with properties at the extreme/optimistic end of the range of expected properties. Model A (1 in paper I) is the standard halo model, which has a Maxwellian local velocity distribution with velocity dispersion equal to $270\,\mathrm{km\,s^{-1}}$. Model B (3) is the logarithmic ellipsoidal model [10], which has a multivariate gaussian velocity distribution, with shape parameters $p = 0.9, q = 0.8$ (corresponding to a density distribution with axis ratios $1 : 0.78 : 0.48$) and velocity anisotropy $\beta = 0.4$. Model C (12) is the standard halo model plus a tidal stream with bulk velocity, in Galactic co-ordinates, $(-65.0, 135.0 - 250.0)\,\mathrm{km\,s^{-1}}$ and velocity dispersion $30\,\mathrm{km\,s^{-1}}$ comprising 25% of the local density [11, 12].

We calculate the recoil distribution for each halo model via Monte Carlo simulation, as described in section IIB of Paper I [4] [1]. In a realistic directional detector it will not be possible to measure the nuclear recoil direction with infinite precision due to multiple scattering and diffusion. In paper I [4] we took these effects into account assuming a TPC detector filled with 0.05 bar CS_2, a 10cm drift length over which a uniform drift field of $1\mathrm{kVcm}^{-1}$ was applied and a 200 μm 3-d pixel readout. Extending these simulations to determine the angular resolution of a 2-d detector is however complicated; projection effects

[1] The recoil momentum spectrum could also be calculated analytically using the radon transform [9], however for our application Monte Carlo simulations would still be required to calculate the distributions of the statistics as a function of number of events.

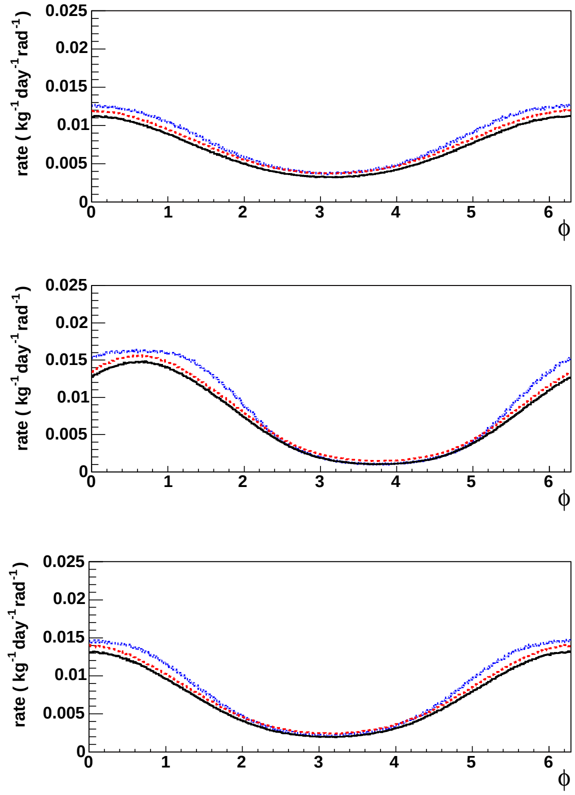

FIG. 1: The distribution of the raw 2-d angles of the recoils in the horizon, meridian and east-west planes from top to bottom for a detector located at Boulby for the three benchmark halo models (from bottom to top at $\phi = 0$): model A standard halo (black solid line), model B triaxial (red dashed), and model C standard halo plus stream (blue dotted). We have set $\sigma_0 = 10^{-6}$ pb and $\rho_0 = 0.3\,\mathrm{GeVcm^{-3}}$ and the integrated distributions give the total event rate per kilogram, per day ($0.043, 0.047$ and $0.050\,\mathrm{kg^{-1}day^{-1}}$ respectively).

combined with multiple scattering of the recoil will make the resolution a function of the primary recoil direction and energy. We therefore assume a 0.05bar CS_2 TPC detector with a 2-d readout of perfect resolution to provide benchmark figures for the numbers of events required for the detection of a WIMP signal with a 2-d detector. The numbers of events we find will hence be lower limits on the numbers required by a real detector. We further take a recoil threshold of 20 keV, as below this energy the recoil tracks are too short for their directions to be reconstructed even in our model 3-d detector.

The orientation of the 2-d readout plane is likely to be determined by the geometry of the lab. We therefore define a Cartesian coordinate system fixed in the laboratory in which the x-axis points towards geographic north, the y-axis towards geographic west and the z-axis towards the zenith. The three simplest possible 2-d read-out planes in this frame are: meridian (x-z plane), horizon (x-y plane) and east-west (y-z plane). We measure 2-d directions by projecting the 3-d recoil vectors into each of these planes

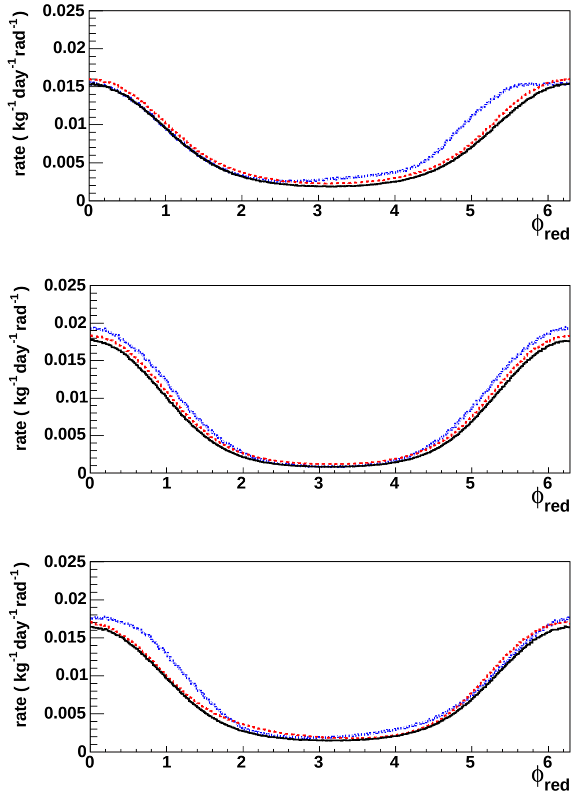

FIG. 2: As Fig. 1 for the reduced angles, with the direction of motion of the Sun (which is a function of time) subtracted.

and measuring the (right-handed) azimuthal angle ϕ between the projected vector and the z-axis in the meridian plane, the x-axis in the horizon plane and the z-axis in the east-west plane. The orientation of the read-out planes also depends on the latitude of the detector. We focus mainly on a detector at the Boulby mine, where the DRIFT detector is currently located, which is at a latitude of 54.5° N. Potential locations are limited by the requirement of a suitable deep underground laboratory (in order to shield cosmic-ray backgrounds). All of the proposed directional detector locations which we are aware of lie at mid-northern latitudes (e.g. Kamioka, SnoLab). To cover the range of possible locations we also examine detectors located at 36.5° N (Kamioka) and 46.1° N (Sudbury).

The distributions of the ϕ angles in each plane for each halo model are generated from the 3-d recoil distributions in the Galactic frame by Monte Carlo simulation of the sidereal time dependent coordinate transform to the detector frame, together with the 2-d projection procedure described above. The size of the anisotropy will be largest if the maximum in the recoil direction distribution a) spends as much time as possible close to the read-out plane (this minimises the spread in the 2-d distribution caused by projection effects) and b) has minimal motion in ϕ (this minimises the spread due to time-averaging). For smooth WIMP distributions the peak recoil direction is the direction of Solar motion and these requirements are fulfilled for any plane whose normal is at 90° to the spin axis of the Earth, or equivalently which contains the Earth's spin axis (see also Ref. [14]). This is the case for all meridian planes and so the anisotropy should be largest in this plane for any location.

In Fig. 1 we plot the raw 2-d angle distributions for the three benchmark halo models for each read-out plane for a detector located at Boulby. For normalisation purposes we have taken the WIMP-nucleon cross section and the local WIMP density to be $\sigma_0 = 10^{-6}\,\mathrm{pb}$ and $\rho_0 = 0.3\,\mathrm{GeV cm^{-3}}$ respectively. As expected the peak-to-trough variation is largest in the meridian plane. The peak-to-trough variation is smallest in the horizon plane as, at Boulby, this plane is furthest from the Earth's spin axis. The standard halo and standard halo with stream (models A and C) have similar peak-to-trough variations, while the triaxial model has a smaller variation (this is not so obvious from the plot, as the three models have different normalisations, reflecting the different event rates above the 20 keV threshold). This suggests that it will be hardest to reject isotropy for read out in the horizon plane and/or the triaxial halo model. The angle distributions at Sudbury and Kamioka are qualitatively similar with the peak-to-trough variation remaining constant in the meridian plane and increasing (decreasing) in the horizon (east-west) plane as the detector location is moved South.

A major difference from the 3-d analysis is that here the recoil directions cannot be transformed from the lab rest frame to the Galactic rest frame. In the 3-d case, the time dependent conversion between the lab and Galactic coordinate frames tends to wash out any anisotropies in lab backgrounds so that they are isotropic in the Galactic rest frame. The modulation of the mean recoil direction with sidereal time (e.g. Ref. [13]) potentially provides a means of checking the Galactic nature of an anisotropic 2-d signal observed in the lab frame. However, determining the mean direction as a function of time necessarily requires large quantities of data. Instead, we use the direction of solar motion projected into each plane, $\mu_{\odot}(t)$, which is sidereal time-dependent in the lab frame, to calculate the reduced angle, $\phi_{\rm red}$, of each recoil:

$$\phi_{\rm red} = \phi - \mu_{\odot}(t)\,, \tag{1}$$

where t is the sidereal time at which the recoil occurred. The time-dependent nature of the reduced angle transform means that any anisotropy in lab backgrounds will be washed out to give isotropic reduced angle distributions.

The reduced angle distributions for the three benchmark halo models for a detector located at Boulby are shown in Fig. 2, and it can be seen that these peak at $\sim 0°$ and have a higher degree of anisotropy compared with the raw distributions in Fig. 1. The reduced angle distributions for model C (with the tidal stream) shows significant excesses with respect to the distributions for

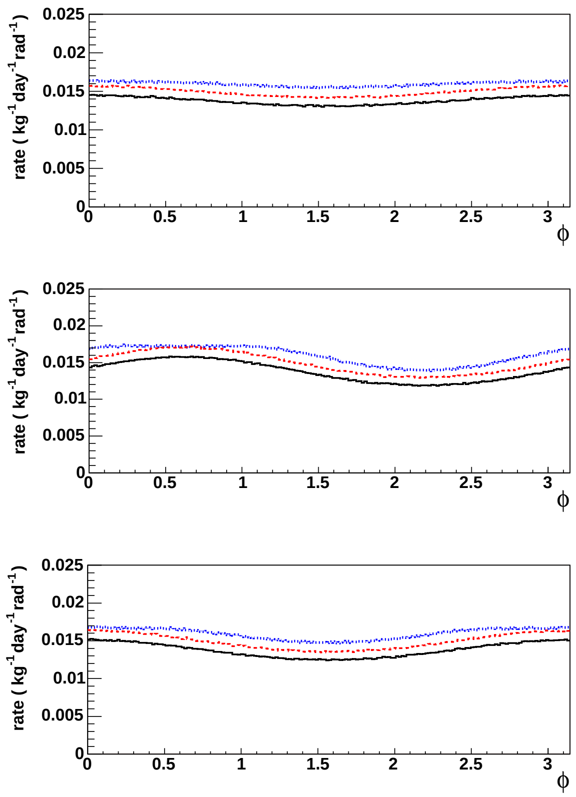

FIG. 3: The raw angle distributions for axial data, where the sense of the recoil can not be measured (halo models, read-out planes and detector location as in Fig. 1).

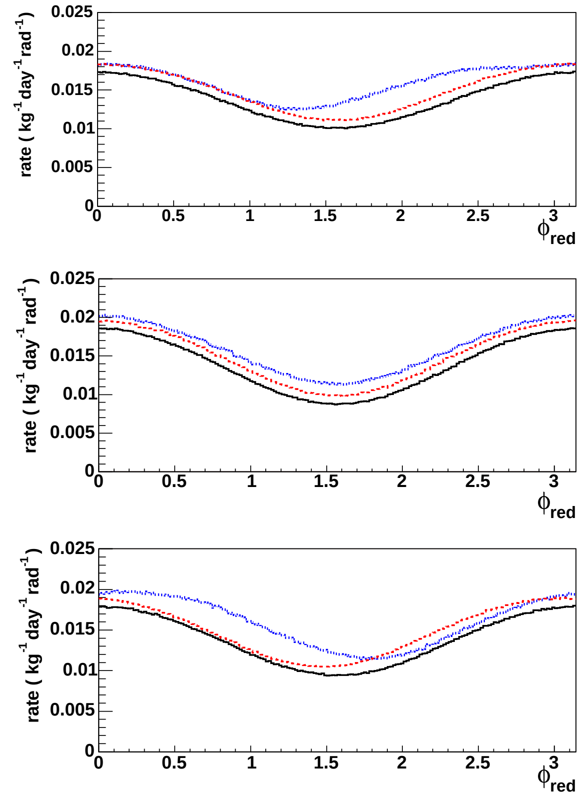

FIG. 4: The reduced angle distributions for axial data, where the sense of the recoil can not be measured (halo models, read-out planes and detector location as in Fig. 1).

the two smooth halo models (at $\phi_{\rm red} \sim 180^\circ - 300^\circ$, $\sim 0^\circ \pm 1^\circ$ and $\sim 0^\circ - 100^\circ$ in the horizon, meridian and east-west planes respectively). The mean reduced angles are $\bar{\phi}_{\rm red} = 348^\circ$, $< 0.1^\circ$ and 4.2° respectively. The deviation of the mean reduced angle from zero remains zero in the meridian plane and increases (decreases) in the horizon (east-west) plane as the detector is moved South. The reduced angle distribution for halo model B has a small asymmetry in the horizon plane at Boulby (the rate at ϕ is smaller (larger) than that at $2\pi - \phi$ for $\phi < (>) \pi/2$). There is a larger deviation from symmetry in the horizon and east-west planes at Sudbury and in the horizon plane at Kamioka. The reduced angle distributions in the meridian plane are always symmetric, as this plane is perpendicular to the flattening of the halo.

It is possible that the absolute signs of the recoil vectors (i.e. their 'sense' $+\vec{x}$ or $-\vec{x}$) may not be measurable, The 2-d raw and reduced angle distribution at Boulby in this case are plotted in Figs. 3 and 4. The peak-to-trough variations of both the raw and reduced angles are significantly smaller than the corresponding vector angle distributions. The anisotropy of the raw axial angle distributions are very small and in particular the direction of the stream in model C is such as to produce an almost flat raw angle distribution in the horizon plane. This is also the case at Kamioka in the east-west plane.

III. APPLYING STATISTICAL TESTS

As we discussed more extensively in Paper I [4] a WIMP search strategy with a directional detector can be divided into three regimes i) search (aiming to detect a non-zero signal), ii) confirmation (does the signal have the anisotropy expected for a Galactic signal?) iii) exploitation (extraction of information about the local WIMP velocity distribution, for instance flattening of the halo or the presence of a tidal stream). We therefore consider three simple hypotheses to test

1. Is the recoil direction distribution uniform?
2. Is the recoil distribution rotationally symmetric about the direction of solar motion?
3. Does the mean direction deviate from the direction of solar motion?

aimed at detecting a WIMP signal, flattening of the MW halo and the presence of a stream of WIMPs respectively.

The recoil directions projected into the read-out plane constitute 2-d vectors, or, if the senses are not known,

undirected lines or axes, and so can equivalently be represented as points on a circle, parameterised by their angle relative to some fixed point/direction. This allows us to use statistical inference methods developed for the analysis of circularly distributed data (for a review of this extensive field see the standard texts such as Refs. [15, 16]). We investigate a variety of non-parametric statistics designed to test the isotropy (Secs. III A), rotational symmetry (Sec. III B) and mean direction (Sec. III C) for the benchmark halo models discussed in Sec. II.

For each halo model and readout plane we calculate the probability distribution of each test statistic, for a given number of events n, by Monte Carlo generating 10^5 experiments each observing n recoils drawn from our calculated ϕ distributions. We then compare this with the null distribution of the statistic, under the assumption of isotropy/rotational symmetry/zero mean reduced angle calculated using analytic expressions where available and otherwise via Monte Carlo simulation. Specifically we calculate the rejection and acceptance factors, R and A, for each value of the statistic. The rejection factor is the probability of measuring a smaller absolute value of the statistic if the null hypothesis is true or equivalently the confidence with which the null hypothesis can be rejected given that measured value of the statistic. The acceptance is the probability of measuring a larger absolute value of the test statistic if the alternative hypothesis is true or equivalently the fraction of experiments in which the alternative hypothesis is true that measure a larger absolute value of the test statistic and hence reject the null hypothesis at confidence level R. Clearly a high rejection factor is required to reject the null hypothesis. We also require a high acceptance, otherwise any one experiment might not be able to reject the null hypothesis at the given rejection factor or the null hypothesis might be erroneously rejected. We therefore find, using a search by bi-section, the number of events required for $A_{\rm c} = R_{\rm c} = 0.9$ and 0.95. For further details on this procedure see Appendix C of Paper I [4].

With 2-d axial data the conventional (but not rigorously justified) procedure is to double the angles of the 2-d vectors and then reduce them modulo 360° (see Sec. A 1 and Ref. [16]) before applying the statistical tests for circular data. This procedure can be used in the case of the isotropy tests but not the symmetry and mean direction tests, as the effect of this transformation on the spread of the reduced angles is not straight forward.

Throughout we assume zero background. This is a reasonable expectation for the next generation of experiments made from low activity materials with efficient gamma rejection and shielding, located deep underground [17]. As with the statistical tests of paper I, non-zero backgrounds can be incorporated into the 2-d tests as described in Section III of paper I. This requires a known background rate, so given that the next generation of directional detectors expect to have essentially zero background, we assume zero background to provide benchmark figures. We should also emphasise that these non-parametric tests, unlike likelihood analyses, do not make any assumptions about the form of the recoil spectrum and can hence be applied to real data (where the underlying local WIMP velocity distribution is unknown).

Halo	$(R_{\rm c}, A_{\rm c}) = (0.90, 0.90)$			$(R_{\rm c}, A_{\rm c}) = (0.95, 0.95)$		
Model	Rayleigh	Kuiper	Watson	Rayleigh	Kuiper	Watson
	Horizon plane					
A	62	70	63	92	103	92
B	70	78	71	103	115	104
C	61	70	62	91	101	91
	Meridian plane					
A	18	20	17	26	29	25
B	21	23	20	30	33	29
C	16	18	16	24	26	23
	East-west plane					
A	28	31	28	41	46	41
B	32	36	32	47	53	47
C	28	31	28	41	46	41

TABLE I: Number of recoil events required to reject isotropy of recoil directions, $N_{\rm iso}$, at 90 (95)% confidence in 90 (95)% of experiments for the Rayleigh, Kuiper and Watson statistics (Appendix A2) for fiducial halo models A (standard), B (triaxial) and C (standard plus stream) for each read-out plane for a detector located at Boulby.

A. Tests of isotropy

We initially focus on a detector located at Boulby. For the Rayleigh, Kuiper and Watson statistics described in Appendix A 2 we determine the minimum number of events required to reject isotropy of recoil directions at 90 (95)% confidence in 90 (95)% of experiments (i.e. for rejection and acceptance probabilities of $R_{\rm c} = A_{\rm c} = 0.9$ and 0.95), $N_{\rm iso}$, for all three benchmark halo models. The results for the raw angle distributions are tabulated in Tables I and II for vector and axial data respectively.

For vector data, in the horizon plane roughly 60 (90) events are required to reject isotropy at 90 (95)% confidence in 90 (95)% of experiments. The numbers of events required in the east-west and meridian planes are smaller by factors of roughly 2 and 3 respectively. For each read-out plane the Rayleigh and Watson tests are equally powerful, with the Kuiper test requiring roughly 10% more events. The triaxial model needs $\sim$ 10% more events than the standard halo model while the tidal stream in model C does not effect the number of events required. These quantitative trends match the expectations from examining the angle distributions in Sec. II. We note that these number are a factor of $2-7$ (depending on the read-out plane) larger than for a 3-d detector with perfect resolution [4].

The same trends (between halo models, read-out

Halo Model	$(R_c, A_c) = (0.90, 0.90)$ Rayleigh	Kuiper	Watson	$(R_c, A_c) = (0.95, 0.95)$ Rayleigh	Kuiper	Watson
	Horizon plane					
A	8 300	9 300	8 400	12 000	14 000	12 000
B	10 000	11 000	10 000	15 000	16 000	15 000
C	> 20000	> 20000	> 20000	> 20000	> 20000	> 20000
	Meridian plane					
A	1 100	1 200	1 100	1 600	1 800	1 600
B	1 200	1 300	1 200	1 800	2 000	1 800
C	1 700	1 900	1 800	2 600	2 800	2 600
	East-west plane					
A	2 100	2 800	2 200	3 100	3 500	3 200
B	2 400	2 800	2 400	3 600	4 000	3 600
C	5 500	5 900	5 400	8 200	8 700	8 000

TABLE II: Same as Table I for axial data (numbers quoted to 2 significant figures).

Halo Model	$(R_c, A_c) = (0.90, 0.90)$ Rayleigh	Kuiper	Watson	$(R_c, A_c) = (0.95, 0.95)$ Rayleigh	Kuiper	Watson
	Horizon plane					
A	21	33	21	30	33	31
B	24	26	24	34	38	25
C	26	29	27	38	42	39
	Meridian plane					
A	12	14	13	18	20	18
B	14	16	14	20	23	21
C	13	14	13	18	20	19
	East-west plane					
A	17	18	17	24	27	25
B	19	21	19	28	31	28
C	19	21	19	28	31	28

TABLE III: Number of events required to reject isotropy of reduced angles.

Halo Model	$(R_c, A_c) = (0.90, 0.90)$ Rayleigh	Kuiper	Watson	$(R_c, A_c) = (0.95, 0.95)$ Rayleigh	Kuiper	Watson
	Horizon plane					
A	310	320	310	450	460	460
B	360	400	360	520	590	530
C	670	740	670	990	1100	990
	Meridian plane					
A	160	180	170	240	270	240
B	200	220	200	290	320	290
C	280	310	280	410	460	410
	East-west plane					
A	220	240	220	320	360	330
B	260	300	260	380	420	380
C	310	350	320	460	510	470

TABLE IV: Same as Table III for axial data (numbers quoted to 2 significant figures).

planes and statistics) are seen for the case of axial data, however the number of events required are increased by roughly 2 orders of magnitude. This is significantly worse than the case of 3-d data, where we found an increase of a single order of magnitude [4]. The increase is even larger for model C in the horizon plane (as noted in Sec. II the direction of the stream in this case is such as to produce an almost flat axial angle distribution).

We also apply the isotropy tests to the reduced angle distributions. The resulting numbers of events required to reject isotropy are given in Tables III and IV for vector and axial data respectively. The variations between halo models and statistics are broadly the same as when the tests are applied to the raw angle distributions, however the numbers of events required are significantly smaller: $\sim 30\%$ in the meridian and east-west planes and a factor of ~ 3 in the horizon plane. The smaller numbers of events reflect the greater anisotropy of the reduced angle distributions.

Halo Model	$(R_c, A_c) = (0.90, 0.90)$ Boulby	Sudbury	Kamioka	$(R_c, A_c) = (0.95, 0.95)$ Boulby	Sudbury	Kamioka
	Horizon plane					
A	21	20	17	30	28	25
B	24	22	19	34	32	28
C	26	24	20	38	35	28
	Meridian plane					
A	12	12	12	18	18	18
B	14	14	14	20	21	20
C	13	13	13	18	18	18
	East-west plane					
A	17	19	21	24	28	30
B	19	21	24	28	31	34
C	19	23	26	28	33	38

TABLE V: Number of recoil events required to reject isotropy of reduced angles using the Rayleigh statistic for each halo model for detectors located at Boulby (54.5° N), Sudbury (46.1° N) and Kamioka (36.5° N).

Tables V and VI compare the numbers of events required to reject isotropy of reduced angles using the Rayleigh statistic for detectors located at Boulby, Sudbury and Kamioka for vectorial and axial data respectively. As expected from the angle distributions, the number of events is smallest (and constant) in the meridian plane and decreases (increases) on moving South for read-out in the horizon (east-west) plane. The, fractional, variation on moving from Boulby to Kamioka is larger for the axial angles, $\sim 35\%$ compared with $\sim 20\%$ for the vectorial angles. The same qualitative trends generally occur for the raw angles but the variations are larger: a factor of ~ 2 (~ 3.4) for vectorial (axial) data. The one exception is model C for axial data. At Sudbury isotropy can be rejected with $< 20\,000$ events in all read-out planes, whereas at Kamioka (Boulby) more than

Halo Model	$(R_c, A_c) = (0.90, 0.90)$ Boulby	Sudbury	Kamioka	$(R_c, A_c) = (0.95, 0.95)$ Boulby	Sudbury	Kamioka
	Horizon plane					
A	310	260	220	450	390	330
B	360	310	260	520	460	390
C	670	340	320	990	490	470
	Meridian plane					
A	160	160	160	240	240	240
B	200	200	200	290	290	290
C	280	280	280	410	410	410
	East-west plane					
A	220	250	300	320	370	440
B	260	300	350	380	440	520
C	310	330	610	460	490	900

TABLE VI: As table V for axial data.

> 20 000 events are required for the east-west (horizon) plane reflecting the close to flat raw axial angle distributions for these location/read-out plane combinations.

Finally, we translate the numbers of events required to reject isotropy with a detector located at Boulby into the equivalent detector exposures, E, required to observe this number of events. If the senses of the recoil directions are observed, isotropy of the reduced angle distribution could be rejected at 95% confidence in 95% of experiments for WIMP-nucleon cross-sections down to $\sigma_0 \sim 7, 4$ and 6×10^{-9} pb for read-out in the horizon, meridian and east-west planes respectively, with an exposure of $E \sim 10^5$ kg day (i.e. a 100 kg detector operating for a period of 2-3 years), assuming a local WIMP density of $\rho_0 \sim 0.3\,\mathrm{GeV\,cm^{-3}}$. For a detector only capable of measuring the recoil axes the sensitivity would be reduced by roughly an order of magnitude. We caution once more that these numbers are for a detector with perfect recoil resolution and hence provide an upper limit on the sensitivity of a real detector. If the isotropy tests were only applied to the raw angle distributions these numbers would be significantly larger, $\sim 50 - 100\%$ for vector data and an order of magnitude for axial data.

B. Test for rotational symmetry

We now examine the number of events which would be required to detect the deviation from rotational symmetry, $N_{\rm rot}$, for models B (triaxial halo) and C (standard halo) using the Wilcoxon signed-rank statistic (Appendix A 3), which is applied to the reduced angle distributions. For both models and all detector locations the reduced angle distributions in the meridian plane are completely symmetric. At Boulby the deviations from symmetry for model B in the other two planes are also small and > 20 000 events would be required. This is also the case for the east-west plane at Kamioka, but for the other location/read-out plane configurations the deviation from symmetry could be detected with of order 5 000 events. The difficulty of detecting flattening of the halo is not surprising as even with 3-d read-out and extremely flattened halo models thousands of events were required [4]. For model C the number of events is strongly dependent on the read-out plane and detector location and reflects the size of the mean reduced angle in each plane. The direction of the stream is close to perpendicular to the direction of Solar motion, so it is not surprising that the plane which is best for detecting isotropy (meridian) is the worst for detecting the stream. For the 10^5 kg day exposure considered in the previous subsection, rotational symmetry could be rejected at 90% confidence down to $\sigma_0 \sim 1, 10$ and $> 50 \times 10^{-7}$ pb for $\rho_0 \sim 0.3\,\mathrm{GeV\,cm^{-3}}$ for model C with a detector located at Boulby. This test is not applicable to axial data.

Halo Model	$(R_c, A_c) = (0.90, 0.90)$ Boulby	Sudbury	Kamioka	$(R_c, A_c) = (0.95, 0.95)$ Boulby	Sudbury	Kamioka
	Horizon plane					
B	>20 000	4 800	5 700	>20 000	7 700	8 900
C	490	1 500	4 500	760	2 300	6 900
	Meridian plane					
B	>20 000	>20 000	>20 000	>20 000	>20 000	>20 000
C	>20 000	>20 000	>20 000	>20 000	>20 000	>20 000
	East-west plane					
B	>20 000	5 800	>20 000	>20 000	9 100	>20 000
C	4 400	1 800	461	6 900	2 800	740

TABLE VII: Number of recoil events required to reject rotational symmetry of recoil directions, $N_{\rm rot}$, at 90 (95)% confidence in 90 (95)% of experiments for the Wilcoxon signed rank statistic (appendix A3) for halo models B (triaxial) and C (standard halo plus stream) and each read-out plane and detector location.

C. Test for mean direction

Finally we use the Watson mean direction test (Appendix A 4) to find the number of events required to detect the deviation from zero of the mean reduced angles for halo model C. The numbers of events required are, for a detector located at Boulby, $\sim 2.5, 10$ and $> 10^2$ times those required with 3-d readout [4] and are similar for the horizon plane and significantly ($\sim 50\%$) smaller in the east-west plane than those required by the rotational symmetry test. This shows that the mean direction test is more powerful than the rotational symmetry test for detecting a tidal stream. For a 10^5 kg day exposure considered in the previous subsection, zero mean reduced angle could be rejected at 90% confidence down to $\sigma_0 \sim 1, 6$ and $> 50 \times 10^{-7}$ pb for $\rho_0 \sim 0.3\,\mathrm{GeV\,cm^{-3}}$ with a detector located at Boulby. This test can not be applied to axial data.

We caution that model C has parameter values at the

Halo Model	$(R_c, A_c) = (0.90, 0.90)$			$(R_c, A_c) = (0.95, 0.95)$		
	Boulby	Sudbury	Kamioka	Boulby	Sudbury	Kamioka
	Horizon plane					
C	470	980	2 500	710	1 500	3 800
	Meridian plane					
C	>20 000	>20 000	>20 000	>20 000	>20 000	>20 000
	East-west plane					
C	2 700	1 300	490	4 200	1 900	750

TABLE VIII: Number of recoil events required to detect a deviation of the mean direction from the direction of solar motion, $N_{\rm dir}$, at 90 (95)% confidence in 90 (95)% of experiments using the Watson mean direction test (appendix A 4) for halo model C (standard halo plus tidal stream) for each read-out plane and detector location.

optimistic ends of the ranges estimated in Ref. [12], i.e. high density and low velocity dispersion. A lower stream density and/or a higher velocity dispersion would give a peak recoil direction closer to the mean direction of motion of the Sun, and make the deviation due to the stream harder to detect. In general the directional detectability of cold streams of WIMPs will clearly depend on how much the projection of their bulk velocity into the read-out plane deviates from the projection of the direction of solar motion

IV. DISCUSSION

We have studied the application of non-parametric tests, developed for the analysis of circular data [15, 16], to the analysis of simulated data as expected from a TPC-based directional WIMP detector with 2-d read-out. As the (energy and direction dependent) resolution of a 2-d directional detector has not been calculated to date we assume perfect resolution. Our results therefore provide a lower limit on the number of events required with a real detector.

We found that if the senses of the recoils are known then between 10 and 30 events, depending on the read-out plane, will be required to reject isotropy of the reduced (with the direction of motion of the Sun subtracted) angle distribution. If the senses are not known then these numbers are increased by roughly an order of magnitude. These numbers are broadly similar to those for full 3-d read-out. If the isotropy tests are applied to the raw angle distribution, however, these numbers increase significantly; $\sim 50\%$ in the meridian and east-west planes and a factor of 3 in the horizon plane for vectorial data. For axial data the increase is even larger, at least an order of magnitude. Using the reduced angle distribution also has the advantage that the time-dependence of the transformation means that even anisotropic lab backgrounds will have isotropic reduced angle distributions. It is therefore crucial that recoil events in a detector with 2-d read-out are accurately time stamped and the reduced angles calculated and analysed. The number of events required is always smallest in the meridian plane, which contains the Earth's spin axis, as for this plane the 2-d projection effects which reduce the size of the isotropy are minimised.

After rejecting isotropy the next step would be to study the direction dependence of the signal and attempt to derive information about the dark matter distribution. If a significant fraction of the local dark matter distribution is in the form of a cold flow/tidal stream then the peak recoil distribution will deviate from the direction of solar motion, or in other words the mean reduced (with the direction of solar motion subtracted) angle will differ from zero. The size of the mean reduced angle (and hence the detectability of the stream) depends on the direction and density of the stream. As an example we consider a stream with bulk velocity, in Galactic co-ordinates, $(-65.0,\, 135.0\, -250.0)\,{\rm km\,s^{-1}}$ comprising 25% of the local density [11, 12]. The number of events required depends on the size of the mean reduced angle and hence the detector location and read out plane. In the meridian plane (which was best for rejecting isotropy) the mean reduced angle is essentially zero, and the stream can not be detected, for all detector locations. For the other read-out planes the number of events ranges between 500 and 5 000. In the horizon (east-west) plane the deviation of the mean reduced angle from zero decreases (increases) as the detector location is moved South from Boulby to Kamioka and the number of events required hence increases (decreases). The number of events required could in theory be reduced by optimising the choice of read-out plane, however in reality this is not feasible due to technical limitations and our lack of knowledge of the underlying WIMP distribution.

It is also potentially interesting to look for deviations from rotational symmetry due to either flattening of the Milky Way halo or the presence of a tidal stream. Unfortunately only a very significant flattening of the Milky Way halo would be detectable. A tidal stream could be detected in this way, however the number of events required is larger than for the mean direction test.

In a realistic directional detector it will not be possible to measure the nuclear recoil direction with infinite precision due to multiple scattering and diffusion. As discussed in Sec. II the angular resolution of a 2-d detector, which will be a function of recoil energy and primary direction due to projection effects, has not yet been calculated. We have therefore assumed a detector with perfect resolution throughout. A rough estimate of the likely degradation in performance due to finite resolution can be obtained by examining the fraction of recoils retaining a sufficiently large ($> 80\%$) fraction of their length after 2-d projection so that their direction can be reconstructed. For the standard halo model 57%, 63% and 60% of the recoils, in the horizon, meridian and east-west planes respectively, retain $> 80\%$ of their length. Therefore the numbers of events required for a realistic

2-d detector with finite direction resolution are likely to be at least a factor of order 2 larger than the numbers we obtain.

In summary, we have found that if the sense of the recoils can be measured, then the potential for detecting a WIMP signal (via its anisotropy) with a detector with 2-d read-out is similar to that for a detector with full 3-d read-out, provided the reduced angle distribution is utilised. If the senses of the recoils can not be measured then the number of events required to detect the anisotropy of a WIMP signal is increased by an order of magnitude, which again is similar to the case of 3-d read out. We should caution, however, that this comparison assumes a detector with perfect resolution. The degradation in performance due to the finite resolution of a real detector might be more significant for 2-d read out than for 3-d read out.

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APPENDIX A: STATISTICAL TESTS FOR CIRCULAR DATA

1. Introduction

We first introduce the necessary basic definitions and terminology relating to circular statistics. For further information see the standard textbooks on this area [15, 16].

2-d vectors are most easily parameterised via their angles ϕ (relative to some arbitrary fiducial direction). Given a sample of n 2-d vectors $\phi_1, ...\phi_n$, if we define

$$C = \sum_{i=1}^{n} \cos\phi_i \,, \tag{A1}$$

$$S = \sum_{i=1}^{n} \sin\phi_i \,, \tag{A2}$$

then the resultant length of the sum of the vectors, R, is given by $R^2 = C^2 + S^2$ and the mean direction $\bar{\phi}$ can be calculated via

$$\bar{\phi} = \arctan\left(S/C\right), \tag{A3}$$

adding π if $C < 0$ and 2π if $S < 0, C > 0$.

The Cartesian coordinates of the centre of mass are denoted by $(\bar{C}, \bar{S})$ where $\bar{C} = C/n$ and $\bar{S} = S/n$, and the mean resultant length is given by $\bar{R} = (\bar{C}^2 + \bar{S}^2)^{1/2}$.

With axial data (i.e. unsigned lines) the standard procedure [16, 18] is to double the axial angles, reduce them modulo 360° and analyse the resulting vector data. There is no rigorous justification for his procedure and it is somewhat limited in its scope [16].

2. Tests of uniformity

The simplest test for uniformity is the Rayleigh test, which uses the mean resultant length, $\bar{R}$, which should be zero, modulo statistical fluctuations, for angles drawn from a uniform distribution. The modified Rayleigh [19] statistic $\mathcal{R}^{\star}$, defined as [15, 20]

$$\mathcal{R}^{\star} = \left(1 - \frac{1}{2n}\right) 2n\bar{R}^2 + \frac{n\bar{R}^4}{2} \,. \tag{A4}$$

has the advantage of approaching its large n asymptotic distribution for smaller values of n than $\bar{R}$. Under the null hypothesis that the distribution from which the sample of angles is drawn is isotropic, $\mathcal{R}^{\star}$ is asymptotically distributed as χ^2_2 with error of order n^{-1} [15]. This test is generally powerful, but is not sensitive to anisotropic distributions with zero mean resultant length (such as antipodally symmetric distributions).

The Kuiper test [15, 21, 22] is a variation of the well-known Kolmogorov-Smirnov test which measures the maximum deviation between the sample cumulative distribution function (cdf) and the cdf of the uniform distribution. The Kuiper test has the advantage of being invariant under cyclic transformations and equally sensitive to deviations between the cdfs over the entire range of ϕ. As in the case of spherical (i.e. 3-d) data the modified Kuiper statistic is defined as

$$\mathcal{V}^{\star} = \mathcal{V}\left(n^{1/2} + 0.155 + \frac{0.24}{n^{1/2}}\right), \tag{A5}$$

where $\mathcal{V}$ is the (unmodified) Kuiper statistic [15, 22]

$$\mathcal{V} = \mathcal{D}^{+} + \mathcal{D}^{-} \,, \tag{A6}$$

and

$$\mathcal{D}^{+} = \max\left(\frac{i}{n} - U_i\right), \quad i = 1, \ldots, n \tag{A7}$$

$$\mathcal{D}^{-} = \max\left(U_i - \frac{i-1}{n}\right), \tag{A8}$$

where $U_i = \phi_i/2\pi$ and the U_i have been ordered so that $U_j \leq U_{j+1}$. An analytic expression for the asymptotic distribution of $\mathcal{V}^{\star}$ under the null hypothesis of uniformity is not available, so we calculate the null distribution numerically via Monte Carlo simulation.

Another alternative test uses Watson's $\mathcal{U}^2$ statistic which measures the mean square deviation between the sample cdf and the cdf of the uniform distribution. The modified U^2 statistic [15, 21] is defined as

$$\mathcal{U}^{\star 2} = \left(\mathcal{U}^2 - \frac{0.1}{n} + \frac{0.1}{n^2}\right)\left(1 + \frac{0.8}{n}\right), \tag{A9}$$

where $\mathcal{U}^2$ is Watson's statistic [15, 23] which can be writ-

ten as

$$\mathcal{U}^2 = \sum_{i=1}^{n}\left(U_i - \bar{U} - \frac{i-1/2}{n} + \frac{1}{2}\right)^2 + \frac{1}{12n}\,, \\ = \sum_{i=1}^{n} U_i^2 - n\bar{U}^2 - \frac{2}{n}\sum_{i=1}^{n} iU_i \\ + (n+1)\bar{U} + \frac{n}{12}\,, \tag{A10}$$

with $\bar{U} = \sum_{i=1}^{n} U_i/n$. As with the Kuiper statistic, the null distribution of $\mathcal{U}^{\star 2}$ has to be calculated numerically.

3. Tests for rotational symmetry

The Wilcoxon signed-rank statistic [24] can be used to test for symmetry about a given direction μ_0. The data is first rotated, i.e. $\phi_i \rightarrow \phi_i = \phi_i - \mu_0$, and the ϕ_i ordered so that $\phi_j < \phi_{j+1}$. The rank of each $|\phi_i|$ amongst $|\phi_1|$, ..., $|\phi_n|$ is calculated and the test statistic $\mathcal{W}_n^{+}$ is given by the sum of the ranks corresponding to $\phi_i > 0$. Any $\phi_i = 0$ should be removed from the data and the sample size n reduced correspondingly, and if more than one ϕ_i has the same absolute value, then they should each be assigned the corresponding average rank.

For $n > 16$

$$\mathcal{W}^{+\star} = \frac{\mathcal{W}_n^{+} - n(n+1)/4 + 0.5}{\left[n(n+1)(2n+1)/24\right]^{1/2}} \tag{A11}$$

is normally distributed [25].

4. Tests for mean direction

A simple test for a given mean direction can be preformed using Watson's $\mathcal{S}$ statistic which directly measures the deviation of the mean of the sampled angles ($\bar{\phi}$) from the hypothesised mean direction (μ_0) [16, 26]. The statistic is defined as

$$\mathcal{S} = \frac{\sin(\bar{\phi} - \mu_0)}{\hat{\sigma}}\,, \tag{A12}$$

where

$$\hat{\sigma} = \frac{1}{\sqrt{2n}\bar{R}}\left[n - \sum_{i=1}^{n}\cos 2(\phi_i - \bar{\phi})\right]^{1/2}\,, \tag{A13}$$

is the sample circular standard error. For $n \geq 25$ $\mathcal{S}$ is normally distributed under the null hypothesis that the sample is drawn from a distribution with mean direction μ_0.

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